Question I

Let A be an $n \times n$ matrix with real and positive eigenvalues and b be a given vector. Consider the solution of $Ax = b$ by the following Richardson's iteration

$$x^{(k+1)} = (I - \omega A)x^{(k)} + \omega b$$

where ω is a damping coefficient. Let λ_1 and λ_n be the smallest and the largest eigenvalues of A. Let $G_\omega = I - \omega A$.

1. Prove that the Richardson's iteration converges if and only if

$$0 < \omega < \frac{2}{\lambda_n}.$$

2. Prove that the optimal choice of ω is given by

$$\omega_{\text{opt}} = \frac{2}{\lambda_1 + \lambda_n}.$$

Prove also that

$$\rho(G_\omega) = \begin{cases} 1 - \omega\lambda_1, & \omega \leq \omega_{\text{opt}} \\ (\lambda_n - \lambda_1)/(\lambda_n + \lambda_1), & \omega = \omega_{\text{opt}} \\ \omega\lambda_n - 1, & \omega \geq \omega_{\text{opt}} \end{cases}$$

where $\rho(G_\omega)$ is the spectral radius of G_ω.

3 Prove that, if A is symmetric and positive definite, then

$$\rho(G_{\omega_{\text{opt}}}) = \frac{\kappa_2(A) - 1}{\kappa_2(A) + 1}$$

where $\kappa_2(A)$ is the spectral condition number of A.

Question II

Let the energy functional for $u(x) \in \mathbb{R}$ ($x \in [0,1]$) be given as

$$E[u] = \int_0^1 \left[\frac{1}{2}|\partial_x u(x)|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 \right] dx, \quad 0 < \varepsilon \ll 1.$$

Consider the dynamical equation (natural boundary condtions)

$$\frac{\mathrm{d}}{\mathrm{dt}} u(x,t) = -\frac{\delta E}{\delta u}[u(x,t)], \quad u(x,0) = u_0,$$

where u_0 is a sufficiently smooth function.

1. Show that $E[u(x,t)]$ is decreasing in t.
2. If initially $u_0(x) \in [-1,1]$, show that $u(x,t) \in [-1,1]$ for all $t > 0$.
3. Design a semi-discrete-in-time scheme such that the energy functional is decreasing for the discrete scheme.

Question III

Let $a_k(t), b_k(t) \in \mathbb{R}$ $(k = 1, 2, \ldots, n)$ satisfy the differential equations:

$$\frac{\mathrm{d}}{\mathrm{dt}} a_k(t) = 2(b_k^2 - b_{k-1}^2), \quad \frac{\mathrm{d}}{\mathrm{dt}} b_k(t) = b_k(a_{k+1} - a_k), \quad k = 1, 2, \cdots, n,$$

where $b_0(t) = b_n(t) = 0$. Consider the $n \times n$ tri-diagonal matrix $L(a,b)$

$$L(a,b) = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & & & 0 \\ & & \ddots & & \\ & & & & b_{n-1} \\ 0 & & & b_{n-1} & a_n \end{bmatrix},$$

show that:

1. The eigenvalues of $L(t) = L(a(t), b(t))$ are independent of t.
2. $\lim_{t \to \infty} b_k(t) = 0$, $k = 1, 2, \cdots, n-1$.